ALGEBRA QUALIFYING EXAM II **MAY 5, 2017**

Note: Explain your answers.

(1) Consider the polynomial $f(x) = x^{15} - 2 \in \mathbb{F}_7[x]$.
   (a) Show that the splitting field of $f(x)$ is generated by a primitive 45th root of unity.
   (b) Determine $n \geq 1$ so that the extension $\mathbb{F}_{7^n}$ contains a splitting field of $f(x)$.
   (c) Compute the degree over $\mathbb{F}_7$ of the splitting field of $f(x)$.

(2) Let p be an odd prime and ζ_p be a primitive pth root of unity.
   (a) Describe $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and determine all primes p such that $\mathbb{Q}(\zeta_p)$ contains a subfield L whose Galois group over $\mathbb{Q}$ is isomorphic to $\mathbb{Z}/5\mathbb{Z}$.
   (b) Using (a), prove that there exists a finite Galois extension E of $\mathbb{Q}$ such that $\mathrm{Gal(E}/\mathbb{Q}) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

(3) (a) Find a projective resolution of $\mathbb{Z}/12\mathbb{Z}$ in the category of abelian groups.
   (b) Compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/9\mathbb{Z})$ for $i \geq 0$.
   (c) Compute $\mathrm{Ext}_{\mathbb{Z}}^i(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/8\mathbb{Z})$ for $i \geq 0$.

(4) Consider the polynomial $f(x) = x^4 + ax^2 + b \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. The roots of this polynoimal are of the form $\pm\alpha, \pm\beta$. Let K be the splitting field of $f(x)$ with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$.
   (a) Show that $|\mathrm{G}| = 4$ or 8.
   (b) Show that if $|\mathrm{G}| = 4$ then only the identity element of G fixes a root of $f(x)$.
   (c) Show that G is the direct product of two cyclic groups of order 2 if and only if b is a square in $\mathbb{Q}$.
   (d) Show that G is cyclic of order 4 if and only if $\frac{a^2-4b}{b}$ is a square in $\mathbb{Q}$.